ZORN'S LEMMA

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1. Introduction

Zermelo gave a beautiful proof in [6] that every set can be well ordered, and Kneser adapted it to give a direct proof of Zorn's lemma in [3]. Sources such as [4], [5], [2, p. 63], and most recently, [1], describe this proof, but it still doesn't seem to be generally known by mathematicians.

2. The proof

A *partially* ordered set is a set X equipped with a relation $x \leq y$ satisfying $x \leq x$ and $x \leq y \leq z \Rightarrow x \leq z$ and $x \leq y \leq x \Leftrightarrow x = y$. (The last property is easily obtained by considering the quotient set for the equivalence relation $x \sim y \Leftrightarrow x \leq y \leq x$.) A *totally* ordered set is a partially ordered set where $x \leq y \vee y \leq x$. A *well* ordered set is a totally ordered set where every nonempty subset has a minimal element. A *closed* subset Y of a partially ordered set X is a subset satisfying $x \leq y \in Y \Rightarrow x \in Y$; we write $Y \leq X$, and if $Y \neq X$, too, then we write $Y < X$. If X is well ordered and $Y < X$, and we take x to be the smallest element of $X - Y$, then $Y = \{y \in X \mid y < x\}$.

Lemma 2.1. *Suppose X is a partially ordered set, and F is a collection of subsets which are well ordered by the ordering of X. Suppose also that for any $C, D \in F$, either $C \leq D$ or $D \leq C$. Let $E = \bigcup_{C \in F} C$. Then E is well ordered, and for each $C \in F$ we have $C \leq E$.*

Theorem 2.2 (Zorn's lemma)**.** *A partially ordered set X with upper bounds for its well ordered subsets has a maximal element.*

Proof. Suppose not. For each well ordered subset $C \subseteq X$ pick an upper bound $g(C) \notin C$. A well ordered subset $C \subseteq X$ such that $c = g(\{c' \in C \mid c' < c\})$ for every $c \in C$ will be called a g-set.

Intuitively, a g-set C, as far as it goes, is determined by g. For example, if C starts out with $\{c_0 < c_1 < c_2 < \dots\}$, then necessarily $c_0 = g(\{\})$, $c_1 = g(\{c_0\})$, $c_2 = g(\{c_0, c_1\})$, and so on. A pseudoproof of the theorem might go like this. We start with an empty collection of g-sets and add larger and larger g-sets to it. At each stage let W be the union of the g-sets encountered previously. We see that $W' = W \cup \{g(W)\}$ is a larger g-set, and we add it to our collection. Continue this procedure forever and let W be the union of the g-sets encountered along the way; it's again a g-set, and we can enlarge it once again, thereby encountering a g-set that isn't in our final collection and providing a contradiction. The problem with this pseudoproof is in interpreting the meaning of "forever", so now we turn to the real proof.

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We claim that if C and D are g-sets, then either $C \leq D$ or $D \leq C$. To see this, let W be the union of the subsets $B \subseteq X$ satisfying $B \leq C$ and $B \leq D$. Since a union of closed subsets is closed, we see that $W \leq C$ and $W \leq D$, and W is the largest subset of X with this property. If $W = C$ or $W = D$ we are done, so assume $W < C$ and $W < D$, and pick elements $c \in C$ and $d \in D$ so that $W = \{c' \in C \mid c' < c\} = \{d' \in D \mid d' < d\}$. Since C and D are g-sets, we see that $c = g(W) = d$. Let $W' = W \cup \{g(W)\}$; it's a g-set larger than W with $W' \leq C$ and $W' \leq D$, contradicting the maximality of W.

Now let W be the union of all the g-sets. It's a g-set, too, and it's the largest g-set, but $W' = W \cup \{g(W)\}$ is a larger g-set, yielding a contradiction. $\square$

References

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