FINITE GENERATION OF K-GROUPS OF A CURVE OVER A FINITE FIELD [AFTER DANIEL QUILLEN]

Daniel R. Grayson*

Columbia University
New York, New York 10027

The main result in this paper was proved by Quillen by 1974. I have used three sources in assembling this exposition: notes taken by Hyman Bass at a talk of Quillen's in Oberwolfach on June 25, 1974; notes from a course given by Quillen at MIT in Spring, 1975; and recent telephone conversations with Quillen.

For a ring of integers in a number field, Quillen had already shown the groups K_iA are finitely generated by September, 1972, when he spoke at the Battelle converence [Q2]. An examination of the proof there reveals that the only portion which is not true for every Dedekind domain A with fraction field F is the following pair of assertions.

(0.1) Pic A *is finite*.

(0.2) *If* P *is a finitely generated projective* A-*module and* $W = P \otimes_A F$, *then* $H_i(Gl(P), st(W))$ *is a finitely generated abelian group for all* i.

Here st(W) denotes the Steinberg module of W. In the notation of [Q2]

*Partially supported by the National Science Foundation.

$\boxed{W}$ is the Tits building of W, and $st(W) = \tilde{H}_{n-2}(\boxed{W})$ with $n = \dim W$.

We phrase this result as follows.

THEOREM 0.3 [Q2]: _If A is a Dedekind domain with fraction field F satisfying_ (0.1) _and_ (0.2), _then_ K_iA _is finitely generated for all_ i.

The title of this paper refers to the following theorem.

THEOREM 0.4: _If C is a nonsingular algebraic curve over a finite field, then_ K_iC _is a finitely generated abelian group for all_ i.

Its proof will of course depend on (0.3). Harder subsequently complemented this result by proving these groups are torsion.

THEOREM 0.5 [H, 3.2.3]: _If C is a nonsingular affine algebraic curve over a finite field, then_ K_iC _is torsion for_ $i > 1$, _and_ SK_1C _is torsion_.

(This is the corect statement of his result, because his techniques apply to Sl_n, not to Gl_n. K_1C is torsion iff C has only one point at infinity, because it contains the units. Bass, Milnor, and Serre [BMS, Corollary 4.3b] have shown, in fact, that $SK_1C = 0$.)

COROLLARY 0.6 [H]: _The groups in_ (0.5) _are finite groups_.

One interesting consequence of Theorem 0.4 is the following.

COROLLARY 0.7: _If A is a finitely generated_ $\mathbb{Z}$-_algebra of dimension_ ≤ 1, _then_ $K_i'A$ _is finitely generated for all_ i.

By definition, $K_i'A = K_i\underline{\underline{M}}(A)$, where $\underline{\underline{M}}(A)$ is the exact category of all finitely generated A-modules.

Proof: If I is a nilpotent ideal in A, then the dévissage theorem [Q1, p.112] implies that $K_i'A = K_i'A/I$; thus we may assume A is reduced. If $f \in A$, then the localization theorem [Q1, p.113] yields an exact sequence:

$$\cdots \longrightarrow K_i'(A/fA) \longrightarrow K_i'(A) \longrightarrow K_i'(A_f) \longrightarrow K_{i-1}'(A/fA) \longrightarrow \cdots \longrightarrow K_0'(A_f) \longrightarrow 0.$$

If f is a nonzerodivisor, then $\dim A/fA < \dim A$, so by induction on dimension we may replace A by A_f. Localizing in this way allows us to assume that the irreducible components of Spec A do not meet. Since K_i' commutes with finite products, we may assume A is a domain. Since all prime fields are perfect, the map Spec $A \to$ Spec $\mathbb{Z}/p$ ($p = \text{char } A \geq 0$) is generically smooth, and we may localize A further to make it smooth. Now A is regular, so the resolution theorem [Q1, p. 110] says that $K_i'A = K_iA$.

If $p = 0$, then A is the ring of S-integers in some number field F for some finite set S of places. Finite generation in this case was proved by Quillen in [Q2].

If $p \neq 0$ and $\dim A = 0$, then A is a finite field, and Quillen computed these explicitly in [Q3]; in particular, K_iA is finitely generated.

If $p \neq 0$ and $\dim A = 1$, then Theorem 0.4 gives the result. Q.E.D.

Remark: Bass has conjectured that $K_0'A$ is finitely generated for any finitely generated $\mathbb{Z}$-algebra. Corollary 0.7 is progress toward the natural generalization of this conjecture to the higher K-groups. Bloch [B] has recently shown $K_0'A$ is finitely generated in the case where char $A = 0$ and $\dim A = 2$, and expects the same techniques to work when char $A \neq 0$ and $\dim A = 2$.

The idea for the proof of Theorem 0.4 arises from Serre's 1968-69 course [S] on Sl_2. He considers a smooth projective curve C over a finite field, a closed point $\infty \in C$, and the coordinate ring A of $C - \{\infty\}$. Serre obtained results about $\Gamma = Gl_2 A$ and its homology by studying the way it acts on the Bruhat-Tits building X for the discrete valuation ring $\mathcal{O}_{C,\infty}$. X is a tree, and he interprets its vertices in terms of vector bundles on C. The vector bundles which are close to being semistable determine a finite subgraph of X/Γ [S, p. 143-6] whose complement is a disjoint union of half lines, one for each element of Pic A. Quillen's contribution here is to extend these ideas to $Gl_n A$.

THEOREM 0.8: *If* Spec A *is a nonsingular affine algebraic curve over a finite field with just one point at infinity, and* F *is its fraction field, then* A *satisfies* (0.2).

The proof of this theorem constitutes the bulk of the remainder of the paper.

Proof of Theorem 0.4 from Theorem 0.8: We may assume C is irreducible because K_i commutes with finite products of rings. Now C is an open subvariety of some projective nonsingular irreducible curve $\bar{C}$. The localization theorem, as used in the proof of (0.7), allows us to replace C by $\bar{C}$. We may then replace C by the complement in C of a single closed point, and let A be its coordinate ring (any algebraic curve, if not projective, is necessarily affine). Theorem 0.8 yields (0.2), and it is easy to check (0.1) using Riemann-Roch. Apply Theorem 0.3. Q.E.D.

1. Preliminaries.

We collect in this section some results about simplicial complexes. Some of them are rather technical, so it seems advisable to separate them from the rest of the proof.

DEFINITION: An ordered simplicial complex is a simplicial complex X with a partial ordering $x \leq x'$ of its set of vertices, Vert(X), which makes each simplex totally ordered.

DEFINITION: Given ordered simplicial complexes X and Y, define their product $X \times Y$ to be the ordered simplicial complex whose set of vertices is the partially ordered set $\text{Vert}(X) \times \text{Vert}(Y)$, and whose simplices are all totally ordered sets of vertices whose projections on X and on Y are simplices.

We use $|\ |$ to denote geometric realization of a simplicial complex, and we give products of CW-complexes the compactly generated topology.

LEMMA 1.1: *The natural map* $f\colon |X \times Y| \to |X| \times |Y|$ *is a homeomorphism.*

Proof: Let $\Delta(m)$ denote the standard simplicial complex with vertices $\{0 < 1 < \ldots < m\}$ (every nonempty subset is a simplex). If $X = \Delta(m)$ and $Y = \Delta(n)$, then f is a homeomorphism [M]. Given simplices in X and Y of dimensions m and n we have natural subcomplexes $\Delta(m) \subset X$ and $\Delta(n) \subset Y$ which preserve the ordering, and thus a subcomplex $\Delta(m) \times \Delta(n) \subset X \times Y$. The diagram

$$\begin{array}{ccc} |\Delta(m) \times \Delta(n)| & \subset & |X \times Y| \\ \Big\downarrow \wr & & \Big\downarrow f \\ |\Delta(m)| \times |\Delta(n)| & \subset & |X| \times |Y| \end{array}$$

is cartesian. Since $|X| \times |Y|$ is covered by such products of simplices, we see that f is a homeomorphism. **Q.E.D.**

DEFINITION 1.2: Suppose $f, g: X \to Y$ are simplicial maps of simplicial complexes, and X is ordered. We call f and g <u>adjacent</u> if, for each simplex σ of X and each $x' \in \sigma$, the set

$$\{f(x) \mid x \in \sigma \ \& \ x \leq x'\} \cup \{g(x) \mid x \in \sigma \ \& \ x \leq x'\}$$

is a simplex of Y.

COROLLARY 1.3: <u>Suppose</u> X <u>is an ordered simplicial complex, and</u> Y <u>is a simplicial complex. Adjacent simplicial maps from</u> X <u>to</u> Y <u>are homotopic</u>.

<u>Proof</u>: Adjacency is precisely the condition required to construct a homotopy after applying (1.1) to $X \times \Delta(1)$. Q.E.D.

DEFINITION: Suppose the group $\mathbb{Z}$ acts on a partially ordered set X. It acts <u>cofinally</u> if for all $x, x' \in X$, there is an $n \in \mathbb{Z}$ with $x + n \geq x'$. We also require $x + 1 > x$ for all x.

DEFINITION 1.4: Suppose $\mathbb{Z}$ acts cofinally on a partially ordered set X. Let $\langle X \rangle$ denote the simplicial complex whose vertices are $\mathbb{Z}$-orbits in X, and where a simplex is any finite nonempty set of vertices whose union is a chain in X.

Identifying a q-simplex of $\langle X \rangle$ with its union, we may regard it as a chain $\ldots < x_i < x_{i+1} < \ldots$ in X (indexed by $\mathbb{Z}$) with $x_i + 1 = x_{i+q+1}$ for all i.

Let $\langle x \rangle$ denote the orbit $x + \mathbb{Z}$. The proof of the following lemma is easy.

LEMMA 1.5: *Suppose* X *and* Y *are as in* (1.4), *and* $f\colon X \to Y$ *is a function such* that

(i) *for* $x \leq x'$ *in* X, $f(x) \leq f(x')$

(ii) *for* x *in* X, $f(x + 1) = f(x) + 1$.

Then the map $\langle f \rangle\colon \langle X \rangle \to \langle Y \rangle$ *defined by* $\langle f \rangle(\langle x \rangle) = \langle f(x) \rangle$ *is a simplicial map*.

DEFINITION 1.6: If X is a $\mathbb{Z}$-poset as in (1.4), then an *augmentation* is a map $\varepsilon\colon X \to \mathbb{Z}$ satisfying the conditions of (1.5). Let X_0 denote the ordered simplicial complex with vertices $\mathrm{Vert}(X_0) = \{x \in X \mid \varepsilon(x) = 0\}$ and whose q-simplices are all sets $\{x_0, \ldots, x_q\}$ of vertices which can be indexed so that $x_0 < \ldots < x_q < x_0 + 1$.

Notice that the natural map $X_0 \to \langle X \rangle$ is an isomorphism of simplicial complexes; however, X_0 is ordered.

DEFINITION 1.7: If X and Y are as in (1.4), $\varepsilon\colon X \to \mathbb{Z}$ is an augmentation, and $f, g\colon X \to Y$ are twomaps satisfying (1.5.i,ii), then we say f and g are *adjacent* if

(i) for x in X, $f(x) \leq g(x)$, and

(ii) for $x < x'$ in X with $\varepsilon(x) < \varepsilon(x')$, we have $g(x) \leq f(x')$.

LEMMA 1.8: *If* f *and* g *are adjacent maps as in* (1.7), *then* $\langle f \rangle$ *and* $\langle g \rangle$ *are homotopic (i.e. their realizations are)*.

Proof: We may compose $\langle f\rangle$ and $\langle g\rangle$ with $X_0 \tilde{\to} \langle X\rangle$, yielding $\langle f\rangle_0$ and $\langle g\rangle_0$. If $\{x_0,\ldots,x_q\}$ is a simplex of X_0 numbered as in (1.6), then $\varepsilon(x_q) = 0 < 1 = \varepsilon(x_0 + 1)$, so by (1.7.ii) $g(x_q) \leq f(x_0 + 1)$. Given i with $0 \leq i \leq q$, we have $f(x_0) \leq\ldots\leq f(x_i) \leq g(x_i) \leq\ldots\leq g(x_q) \leq f(x_0) + 1$, so $\langle f\rangle_0$ and $\langle g\rangle_0$ are adjacent maps of simplicial complexes as in (1.2), and we may apply (1.3). Q.E.D.

This ends our discussion of these matters. We need one more result about nerves of coverings. Segal has recorded the result for open coverings in [Se]--we need the same result for closed coverings.

LEMMA 1.9: Suppose X is a simplicial complex, T is a poset, and for each $\sigma \in T$ we are given a subcomplex X_σ of X. Suppose the following properties hold.

(i) $\sigma \leq \tau$ implies $X_\sigma \supset X_\tau$

(ii) $X = \cup X_\sigma$ (i.e. every simplex of X is a simplex of some X_σ)

(iii) each X_σ is contractible

(iv) for each simplex γ in X the poset $T_\gamma = \{\sigma \mid \gamma$ is a simplex of $X_\sigma\}$ is contractible.

Then X and T are homotopy equivalent in a natural way.

Proof: We follow the notation of [Q4, p. 103-4]. One checks that (i)-(iv) remain true if X and each X_σ are replaced by their barycentric subdivisions (poset of simplices); Thus we may assume W is a poset and each X_σ is a closed subset. Consider the incidence correspondence $Z = \{(x,\sigma) \in X \times T \mid x \in X_\sigma\}$; it is a closed subset of $X \times T$. Now [Q4, Corollary 1.8] applies because for each $\sigma \in T$, $Z_\sigma = X_\sigma$ is contractible, and for each $x \in X$, $Z_x = T_x$ is contractible. Q.E.D.

2. The Bruhat-Tits Building.

Fix a discrete valuation ring R, its fraction field F, a uniformizing parameter π, and an F-vector space W of dimension n.

The group $Gl(W)$ acts naturally on the poset $\underline{\underline{L}} = \underline{\underline{L}}(W)$ of all R-lattices in W. The group $\mathbb{Z} = F^{\times}/R^{\times}$ acts naturally on $\underline{\underline{L}}$ by homothety so that $L + n = \pi^{-n}L$ for $L \in \underline{\underline{L}}$. This action is cofinal. The Bruhat-Tits building $X = X(W)$ is defined to be the simplicial complex $\langle\underline{\underline{L}}\rangle$ introduced in (1.4). Since any lattice L has $L/\pi L$ of length n, we see that $\dim X = n - 1$.

Since $\underline{\underline{L}}(F) = \mathbb{Z}$, an augmentation (1.6) $\varepsilon: \underline{\underline{L}}(W) \to \mathbb{Z}$ can be obtained by choosing a surjective F-linear map $W \to F$. Another augmentation comes from the index: $\varepsilon(L) = \lceil(\mathrm{ind}(L,L_0))/n\rceil$, where L_0 is a fixed lattice. Here $\mathrm{ind}(L,L_0) = \mathrm{length}\ (L/L_1) - \mathrm{length}\ (L_0/L_1)$ for any lattice L_1 contained in L and L_0.

THEOREM 2.1 [BT]: X <u>is contractible</u>.

<u>Proof</u>: Let $\varepsilon: \underline{\underline{L}} \to \mathbb{Z}$ be an augmentation. Fix $\Lambda \in \underline{\underline{L}}$ and for each $n \in \mathbb{Z}$ define $F_n: \underline{\underline{L}} \to \underline{\underline{L}}$ by

$$F_n(L) = L + (\pi^{-1})^{n+\varepsilon(L)}\Lambda.$$

It is easy to see that F_n satisfies (1.5.i,ii) and that F_n and F_{n+1} are adjacent (1.7). Thus by (1.8) $\langle F_n\rangle$ and $\langle F_{n+1}\rangle$ are homotopic maps from X to X. For $L \in \underline{\underline{L}}$, we see that

$$\langle F_n(L)\rangle = \begin{cases} \langle L\rangle & n \ll 0 \\ \langle\Lambda\rangle & n \gg 0\,. \end{cases}$$

If $f\colon Z \to |X|$ is a continuous map from a compact space Z, then $f(Z)$ is carried by a finite number of vertices of X, and thus

$$|\langle F_n \rangle| \circ f \begin{cases} = f & n \ll 0 \\ \text{is constant} & n \gg 0. \end{cases}$$

This shows that $\pi_i|X| = 0$ for $i \geq 0$, and we conclude from a theorem of Whitehead that $|X|$ is contractible. Q.E.D.

Note: Considering the unit interval as a two-point compactification of $\mathbb{R}$ provides an explicit contraction of $|X|$, so the appeal to Whitehead's theorem is not needed.

3. Stable vector bundles

In this section we review the basic facts about stable vector bundles on a nonsingular irreducible projective curve C over a field k. Set also [NS, Section 4] and [HN, Section 1.3].

We do not require k to be algebraically closed. Let $F = k(C)$ be the function field of C.

A vector bundle on C is a locally free sheaf of $\mathcal{O}_C$-modules of finite rank. Any quasi-coherent subsheaf E_1 of a vector bundle E is also a vector bundle, and is called a subbundle if E/E_1 is a vector bundle. Given subbundles $E_1 \subset E_2$ of E, it follows that E_1 is also a subbundle of E_2.

If W' is an F-subspace of $E \otimes F$, then $E \cap W'$ is a subbundle of E; if E' is a subbundle of E, then $E' \otimes F$ is an F-subspace of $E \otimes F$.

These two operations set up a one-to-one correspondence between subbundles E' of E and subspaces W' of $E \otimes F$.

Every subsheaf $E_1 \subset E$ is contained in a unique subbundle $\bar{E}_1 \subset E$ of the same rank, namely, $\bar{E}_1 = E \cap (E_1 \otimes F)$.

The slope of a nonzero vector bundle is defined to be $\mu(E) = (\deg E)/(\operatorname{rank} E)$. The additivity of degree and rank over short exact sequences makes the term "slope" opposite because $\mu(E_1/E_2)$ is the slope of the line joining the points (in the rank-degree plane) corresponding to E_1 and E_2. We also have the formulas

$$\mu(E_1 \otimes E_2) = \mu(E_1) + \mu(E_2)$$

$$\mu(E^\vee) = -\mu(E).$$

A vector bundle E is called <u>stable</u> (resp. <u>semistable</u>) if for all nonzero subbundles $E_1 \subset E$ (or for all subsheaves) we have $\mu(E_1) < \mu(E)$ (resp. $\leq$). An <u>unstable</u> bundle is one which is not semistable.

Stability can also be described in terms of quotient bundles of E, because $\mu(E_1) < \mu(E)$ iff $\mu(E/E_1) > \mu(E)$.

One sees that the degrees of subbundles of E are bounded above by intersecting a subbundle with a fixed flag $0 \subset E_1 \subset \ldots \subset E_n = E$ of subbundles with rank $E_i = i$. Thus the slopes of subbundles of E are discrete and bounded above, and there exist subbundles of maximum slope. We let $\mu_{max}(E)$ denote the maximum slope of a subbundle of E; we let $\mu_{min}(E)$ denote the minimum slope of a quotient bundle of E. We see that:

$$\mu_{min} E = -\mu_{max}(E^\vee).$$

LEMMA 3.1: *Suppose* $E \subset E'$ *are vector bundles of the same rank. Then the following formulas hold.*

(i) $\mu(E) \leq \mu(E')$

(ii) $\mu_{max}(E) \leq \mu_{max}(E')$

(iii) $\mu_{min}(E) \leq \mu_{min}(E')$

Proof: The first assertion follows directly from the additivity of degree and the fact that $\deg(E'/E) \geq 0$. Now if W' is a subspace of $E \otimes F = E' \otimes F$, then $E \cap W' \subset E' \cap W'$ and $E/E \cap W' \subset E'/E' \cap W'$, so (i) yields (ii) and (iii). Q.E.D.

LEMMA 3.2: *If* E_1 *and* E_2 *are semistable vector bundles on* C, *and* $\mathrm{Hom}(E_1,E_2) \neq 0$, *then* $\mu(E_1) \leq \mu(E_2)$.

Proof: Given $f: E_1 \to E_2$ nonzero, it factors as a composite $E_1 \twoheadrightarrow E_3 \subset \overline{E_3} \subset E_2$, where $\overline{E_3}$ is a subbundle of E_2. Then $\mu(E_1) \leq \mu(E_3) \leq \mu(\overline{E_3}) \leq \mu(E_2)$. Q.E.D.

PROPOSITION 3.3 [HN]: *A vector bundle* E *on* C *has a unique flag of subbundles* $0 = E_0 < E_1 < \ldots < E_r = E$ *satisfying the following two properties.*

(i) E_i/E_{i-1} *is semistable for each* i

(ii) $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$, *for each* i.

Moreover, this flag also satisfies

(iii) E_i/E_{i-1} *is the largest subbundle of* E/E_{i-1} *with slope equal to* $\mu_{max}(E/E_i)$

(iii') E_i/E_{i-1} *is the largest quotient bundle of* E_i *with slope equal to* $\mu_{min}(E_i)$.

Proof: The first assertion is exactly [HN, Lemmas 1.3.7,8] together with the observation that the proof does not use their assumption that k is algebraically closed. Now we show (iii): let E' be a subbundle of E/E_{i-1} with $\mu(E') = \mu_{max}(E/E_{i-1})$; it is enough to show that $E' \subset E_i$. Clearly E' is semistable and $\mu(E') > \mu(E_j/E_{j-1})$ for all $j > i$, using (3.2) and descending induction we see that $E' \subset E_{j-1}$ for all $j > i$. This proves (iii), and (iii') follows by applying (iii) to the dual vector bundle of E. Q.E.D.

We will call the flag $E_1 < \ldots < E_{r-1}$ of E from the previous proposition the <u>canonical filtration</u> of E. We let S(E) denote the corresponding flag $E_1 \otimes F < \ldots < E_{r-1} \otimes F$ in $W = E \otimes F$, and will also call it the canonical filtration of E.

The following corollary tells when S(E) can be deduced from the canonical filtrations for a subbundle of E and for the corresponding quotient bundle.

COROLLARY 3.4: <u>Suppose</u> E' <u>is a subbundle of</u> E, $\mu_{max}(E/E') < \mu_{min}(E')$, $E'_1 < \ldots < E'_{r-1}$ <u>is the canonical filtration of</u> E, <u>and</u> $E_1/E' < \ldots < E_{s-1}/E'$ <u>is the canonical filtration of</u> E/E'. <u>Then</u> $E'_1 < \ldots < E'_{r-1} < E' < E_1 < \ldots < E_{s-1}$ <u>is the canonical filtration of</u> E.

Now fix an invertible sheaf $\mathcal{O}(1)$ on C, and adopt the usual notation: $E(m) = E \otimes \mathcal{O}(1)^{\otimes m}$. We declare two vector bundles E_1 and E_2 to be <u>equivalent</u> if for some m there is an isomorphism $E_1 \simeq E_2(m)$, and consider vector bundle classes.

We assume that $\mathcal{O}(1)$ has positive degree. It follows from the representability of the moduli space for semistable vector bundles that there are only finitely many semistable vector bundle classes of rank n; we will need something slightly stronger, and we prove it directly.

Define $\mu_{diff}(E) = \mu_{max}(E) - \mu_{min}(E)$, and notice that it depends only on the class of E.

PROPOSITION 3.5: *If k is a finite field, then given integers n and N, there are only finitely many vector bundle classes E with rank $E = n$ and $\mu_{diff}(E) \leq N$.*

Proof: For each class we may choose a representative E with $g - 1 < \mu_{min}(E) \leq g - 1 + e$ (where $e = \deg \mathcal{O}(1)$) because $\mu_{min}(E(m)) = \mu_{min}(E) + me$. Thus $\mu_{max}(E) \leq N + g - 1 + e$. Every quotient bundle E/E' has $\mu(E/E') \geq \mu_{min} > g - 1$, so by the Riemann-Roch theorem E/E' has a nonzero global section, and thus has a rank 1 subbundle with a section. Using this fact and induction allows us to construct a flag $0 = E_0 < E_1 < \ldots < E_n = E$ with rank $E_i = i$ and each E_i/E_{i-1} being the line bundle of an effective divisor. It follows that $\deg E_{i-1} \geq 0$, and $\deg E_i/E_{i-1} = \deg E_i - \deg E_{i-1} \leq \deg E_i = i\mu(E_i) \leq n\mu_{max}(E) \leq n(N + g - 1 + e)$. Since there are only a finite number of points on C of any given degree, we see that the line bundles E_i/E_{i-1} are drawn from a finite set of isomorphism classes (a set depending only on N and n).

Extensions of bundles E' and E'' are classified by the group $\mathrm{Ext}^1(E'',E') = H^1(C, E' \otimes E''^{\vee})$, which is a finite dimensional vector space over k, and thus is a finite set. Since E is built up by successive extension from the linebundles E_i/E_{i-1}, we see that, up to isomorphism, there are only a finite number of possibilities for E.

Q.E.D.

Remark: E is semistable iff $\mu_{diff}(E) \leq 0$ iff $\mu_{diff}(E) = 0$.

4. Stability and the Building.

We preserve the notation C, k, and F from the previous section. Let ∞ be a closed point of C. The open set $U = C - \infty$ is affine, so let A be its coordinate ring. Let $R = \mathcal{O}_{C,\infty}$ be the local ring at ∞, and choose a uniformizing parameter π for R.

Let P be a finitely generated A-module, $W = P \otimes_A F$, $n = \dim_F W$, and $\Gamma = \mathrm{Aut}(P) \subset \mathrm{Gl}(W)$. Let $\tilde{W}$ be the constant sheaf on C associated to W. Let $\mathcal{O}(1) = \mathcal{O}(\infty)$.

DEFINITION: $\underline{\underline{E}}(P)$ denotes the poset of all coherent locally free subsheaves E of $\tilde{W}$ such that $E|_U = P$. Let $\mathbb{Z}$ act on $\underline{\underline{E}}(P)$ via $E + n = E(n)$.

Notice that Γ acts naturally on $\underline{\underline{E}}(P)$.

If $L \in \underline{\underline{L}}(W)$ (see section 2), then there is a unique vector bundle $E(P,L) \in \underline{\underline{E}}(P)$ such that $E(P,L)|_{\mathrm{Spec}\ R} = L$. Thus we have an isomorphism

$$\underline{\underline{L}}(W) \cong \underline{\underline{E}}(P)$$

which is order preserving, Γ-equivariant, and $\mathbb{Z}$-equivariant. We define $X = X(P) = \langle \underline{\underline{E}}(P) \rangle$; it is isomorphic to $X(W)$, the building defined in section 2.

The Tits building $\boxed{W}$ is the poset of subspaces $W' \subset W$ with $0 \neq W' \neq W$. We use $\mathrm{Simpl}\,\boxed{W}$ to denote the poset of chains (simplices) of $\boxed{W}$.

The canonical filtration of section 3 defines a Γ-equivariant function

$$S: \mathrm{Vert}\ X(P) \longrightarrow \mathrm{Simpl}\ \boxed{W} \cup \{\varphi\}$$

$$\langle E\rangle \longmapsto S(E),$$

because $S(E(m)) = S(E)$.

DEFINITION: Given $\sigma \in \mathrm{Simpl}\boxed{W} \cup \{\varphi\}$ let $\underline{\underline{E}}(P)_\sigma = \{E \in \underline{\underline{E}}(P) \mid \sigma \subset S(E)\}$ and let $X_\sigma = X(P)_\sigma = \langle\underline{\underline{E}}(P)_\sigma\rangle$. (Notice that $\mathbb{Z}$ acts on $\underline{\underline{E}}(P)_\sigma$, too, because $S(E(m)) = S(E)$). Let $X' = \bigcup_{\sigma\neq\emptyset} X_\sigma$.

Notice that the vertices of $X - X'$ are those $\langle E\rangle$ with E being a semistable vector bundle.

THEOREM 4.1: <u>Each X_σ is contractible</u>.

<u>Proof</u>: Say $\sigma = \{W_0 <\ldots< W_q\}$. Let $\tau = \{W_1/W_0 <\ldots< W_q/W_0\}$, and let $P_0 = P \cap W_0$. By induction on cardinality of σ we may assume that $X(P/P_0)_\tau$ is contractible, the case when $\tau = \emptyset$ being Theorem 2.1. There is a natural map

$$\alpha\colon \underline{\underline{E}}(P)_\sigma \longrightarrow \underline{\underline{E}}(P/P_0)_\tau$$

defined by $\alpha(E) = E/E \cap W_0$; this map satisfies (1.5.i,ii), and has a section, which we proceed now to define.

Choose $E_0 \in \underline{\underline{E}}(P_0)$ so that $\mu_{min}E_0 > 0$, and choose a splitting $P \cong P_0 \oplus P/P_0$.

Define, for any vector bundle E', an integer $\varepsilon(E') = \lceil(\mu_{max}E')/e\rceil$; here $e = [k(\infty):k]$. In this way we get augmentations (see (1.6)) $\varepsilon\colon \underline{\underline{E}}(P) \to \mathbb{Z}$ and $\varepsilon\colon \underline{\underline{E}}(P/P_0) \to \mathbb{Z}$.

Define $\beta\colon \underline{\underline{E}}(P/P_0)_\tau \to \underline{\underline{E}}(P)_\sigma$ by setting $\beta(E') = E_0(\varepsilon(E')) \oplus E'$; the splitting we chose for P tells how to regard $\beta(E')$ as a subsheaf of $\widetilde{W}$. The map β satisfies (1.5.i,ii).

Let's check that the target of β is as claimed, so suppose

$\tau \subset S(E')$--we must show $\sigma \subset S(\beta(E'))$. We use (3.4) and compute

$$\begin{aligned}\mu_{min}(\varepsilon(E') \cap W_0) &= \mu_{min}(E_0(\varepsilon(E'))) \\ &= \mu_{min}(E_0) + e\cdot\varepsilon(E') \\ &> \mu_{max}E' \\ &= \mu_{max}(\beta(E')/\beta(E') \cap W_0).\end{aligned}$$

Now it is also clear that $\alpha\circ\beta = 1$.

Define for each $n \in \mathbb{Z}$ a map $G_n\colon \underline{\underline{E}}(P)_\sigma \to \underline{\underline{E}}(P)_\sigma$ by setting $G_n(E) = E + E_0(n + \varepsilon(E))$; it is order preserving and $\mathbb{Z}$-equivariant (1.5). We check now that the target of G_n is as claimed, so suppose $\sigma \subset S(E)$--we show $\sigma \subset S(G_n(E))$. Firstly, letting $G = G_n(E)$, $G/G \cap W_0 = E/E \cap W_0$ and $G \cap W_0 \supset E \cap W_0$. Thus $\mu_{min}G \cap W_0 \geq \mu_{min}E \cap W_0 > \mu_{max}E/E \cap W_0 = \mu_{max}G/G \cap W_0$. Now use (3.4).

It is easy to check that G_n and G_{n+1} are adjacent (1.7), and thus $\langle G_n\rangle$ and $\langle G_{n+1}\rangle$ are homotopic (1.8). Notice that $\alpha G_n = \alpha$. For any $E \in \underline{\underline{E}}(P)_\sigma$ we see that

$$G_n(E) = \begin{cases} E & n \ll 0 \\ G_n\beta\alpha(E) & n \gg 0\,. \end{cases}$$

We use G_n just as F_n was used in the proof of (2.1). Let $f\colon Z \to |X(P)_\sigma|$ be any map from a compact space Z. Since $f(Z)$ is carried by a finite number of vertices, we see that

$$|\langle G_n\rangle|\circ f = \begin{cases} |\langle G_n\rangle|\circ|\langle\beta\rangle|\circ|\langle\alpha\rangle|\circ f & n \gg 0 \\ f & n \ll 0. \end{cases}$$

Thus f and $|\langle\beta\rangle| \circ |\langle\alpha\rangle| \circ f$ are homotopic. Since $X(W/W_0)_\tau$ is contractible by induction, we see that $|\langle\alpha\rangle| \circ f$ is null-homotopic, and thus f is, too. This shows that $X(P)_\sigma$ is contractible. Q.E.D.

COROLLARY 4.2: <u>There is a Γ-equivariant homotopy equivalence</u> $|X'| \simeq |\boxed{W}|$.

<u>Proof</u>: We apply (1.9) with $X = X'$, $X_\sigma = X(P)_\sigma$, and $T = \mathrm{Simpl}\boxed{W}$. Property (1.9.iv) holds because $T_\gamma = \{\sigma \mid \sigma \subset S(E)$ for each vertex $\langle E\rangle$ of $\gamma\}$ has a maximal element, namely, $\cap\, S(E)$, and is thus contractible. Now use the natural homotopy equivalence between $\boxed{W}$ and $\mathrm{Simpl}\boxed{W}$. Q.E.D.

THEOREM 4.3: <u>There are only a finite number of Γ-orbits of simplices in</u> $X - X'$.

<u>Proof</u>: Notice that a simplex ξ of $X - X'$ may have none of its vertices in $X - X'$; it is enough that its vertices are not all in the same X_σ. Since any two vertices of a simplex are adjacent, we see that each vertex $\langle E\rangle$ of ξ has the property

(*) for any $W' \in S(E)$, there is a vertex $\langle E'\rangle$ adjacent to $\langle E\rangle$ with $W' \notin S(E')$.

Forgetting ξ, it will be enough to show there are, mod Γ, only finitely many such vertices $\langle E\rangle$.

With W' and E' as in (*), we may assume $E \subset E' \subset E(1)$, and compute

$$\mu_{min} E \cap W' \leq \mu_{min} E' \cap W'$$

$$\leq \mu_{max} E'/E' \cap W'$$

[use (3.4)]

$$\leq \mu_{max}((E/E \cap W')(1))$$

$$= \mu_{max} E/E \cap W' + e.$$

Here $e = [k(\infty):k]$. Thus each slope change in the canonicalfiltration of E is not more than e, so E satisfies

$$\mu_{max}(E) \leq \mu_{min}(E) + e(n - 1).$$

Two vertices $\langle E_1\rangle$ and $\langle E_2\rangle$ are in the same Γ-orbit iff $E_1 \cong E_2(m)$ for some m; we conclude by applying (3.5). Q.E.D.

5. Homology Computations.

We preserve the notation from section 4. Let $n = \dim W$; we have the following theorem of Solomon and Tits.

THEOREM 5.1 [Q2]: *If* $n \geq 2$, *then* $\boxed{W}$ *has the homotopy type of a bouquet of* $(n-2)$-*spheres*.

The Steinberg module, $st(W)$, is $\tilde{H}_{n-2}(\boxed{W}, \mathbb{Z})$ together with the natural action of $Gl(W)$ on it. For $n = 1$, $st(W)$ is $\mathbb{Z}$ with $Gl(W)$ acting trivially. For $n \geq 1$, we see that $st(W) = H_{n-1}(S\boxed{W})$, where S denotes suspension.

We are now in a position to prove the main theorem from the introduction.

Proof of Theorem 0.8: If $x = \langle E\rangle$ is a vertex of $X = X(P)$, and Γ_x is the stabilizer, then it is easy to see that $\Gamma_x = Aut(E)$; this group is finite because it is contained in the finite dimensional k-vector space $\mathrm{End}(E) = H^0(C, E \otimes E^\vee)$.

By Theorem 4.3, there are only a finite number of Γ-orbits of vertices occurring in simplices of $X - X'$. The group $\Gamma = \mathrm{Aut}_A(P)$ is residually finite[1] because all nontrivial quotient rings of A are finite, so we may find a normal subgroup $\Gamma' \lhd \Gamma$ of finite index which acts freely on the simplices of $X - X'$.

Suppose now that $n \geq 2$. For the relative homology we combine (4.2) and (2.1) to get

$$H_i(X,X') = \tilde{H}_i(S\boxed{W}) = \begin{cases} 0 & i \neq n-1 \\ \mathrm{st}(W) & i = n-1. \end{cases}$$

Let $C_q = C_q(X,X')$ be the group of relative chains (isomorphic to the free abelian group on q-simplices of $X - X'$). Since X has dimension $n - 1$, the homology computation yields an exact sequence of Γ-modules.

$$0 \longrightarrow \mathrm{st}(W) \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0.$$

Since each C_q is a finitely generated free $\mathbb{Z}\Gamma'$-module we see that $\mathrm{st}(W)$ is a finitely generated projective $\mathbb{Z}\Gamma'$-module, so

$$H_i(\Gamma', \mathrm{st}(W)) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z}^a & i = 0, \text{ some } a. \end{cases}$$

In particular, $H_i(\Gamma', \mathrm{st}(W))$ is finitely generated for all i. Now the spectral sequence

---

[1] A group Γ is called <u>residually finite</u> if every nontrivial element of Γ maps nontrivially to some finite quotient group of Γ.

$$H_p(\Gamma/\Gamma', H_q(\Gamma', st(W)) \Rightarrow H_{p+q}(\Gamma, st(W))$$

and the fact that Γ/Γ' is finite yield the finite generation of $H_i(\Gamma, st(W))$ for all i.

The case when $n = 1$ is trivial because then $\Gamma = Gl_1(A) = A^\times$ is a finite group. Q.E.D.

REFERENCES

[B] S. Bloch, Algebraic K-theory and Class field theory for Arithmetic Surfaces, preprint.

[BMS] H. Bass, J. Milnor, J.-P. Serre, Solution of the Congruence Subgroup Problem for Sl_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$, I.H.E.S. Publ. Math. 33 (1967) 59-137.

[BT] F. Bruhat and J. Tits, Groupes reductifs sur un corps local. I. Données radicielles valuées. Publ. Math. I.H.E.S. 41 (1972) 5-251.

[H] G. Harder, Die Kohomologie S-arithmetischer Gruppen uber Funktionenkörpern, Inventiones Math. 42 (1977) 135-175.

[HN] G. Harder, M.S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, Math. Annalen 212 (1975) 215-248.

[M] J. Milnor, The realizationof a semi-simplicial complex, Annals of Math. 65 (1957) 272-280.

[NS] M.S. Narasimhan, C.S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Annals of Math. 82 (1965) 540-567.

[Q1] D. Quillen, Higher algebraic K-theory: I, in "Algebraic K-theory I", Lecture Notes in Math. #341, Springer-Verlag, Berlin (1973) 77-139.

[Q2] D. Quillen, Finite generation of the groups K_i of rings of algebraic integers, same volume, 195-214.

[Q3] D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Annals of Math. 96 (1972) 552-586.

[Q4] D. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, Advances in Math., 28 (1978) 101-128.

[Se] G. Segal, Classifying Spaces and Spectral Sequences, Publ. Math. I.H.E.S. 34 (1968) 105-112.

[S] J.-P. Serre, Arbres, Amalgames, Sl_2, Soc. Math. de France, Asterisque #46 (1977), Paris.