Finite generation of the groups K_i of rings of algebraic integers

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§1 Statement of results

The title refers to:

THEOREM 1: *If* A *is the ring of algebraic integers in a number field* F *(finite over* $\mathbb{Q}$*) then* K_iA *is a finitely generated group for all* $i \geq 0$.

Remarks: (1) The proof uses the definition of the groups K_iA, as $K_i\underline{\underline{P}}(A)$, given in [Quillen 1, §2]. Here $\underline{\underline{P}}(A)$ denotes the "exact category" of finitely generated projective A-modules.

(2) If B is the ring of S-integers relative to some finite set S of finite primes of F then we have the localisation sequence [Quillen, 1, §5, Cor. to Thm. 5],

$$\cdots \longrightarrow K_iA \longrightarrow K_iB \longrightarrow \coprod_{\mathfrak{p} \in S} K_{i-1}(A/\mathfrak{p}) \longrightarrow K_{i-1}A \longrightarrow \cdots$$

From [Quillen, 2, Thm. 8] one knows the K-groups of the finite

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fields $A/\mathfrak{p}$, whence one concludes that $K_iA \to K_iB$ has finite kernel (which is zero for i even, > 0) and finite cokernel for $i > 1$. In particular K_iB is finitely generated, and of the same rank as K_iA for $i \neq 1$.

(3) The ranks of the groups K_iA have been computed in [Borel, Cor. of Thm. 2]. Conjectures about the arithmetic significance of the torsion subgroups of the K_iA can be found in [Lichtenbaum].

(4) The proof of Theorem 1 yields the analogous result for a maximal order in a semi-simple F-algebra (cf. the end of §3).

To explain how Theorem 1 is deduced we must introduce the building of a vector space V, say of dimension n, over a field (or a division ring) F. It is the simplicial complex, here denoted $\boxed{V}$, associated to the set of proper subspaces W of V $(0 < W < V)$, (partially) ordered by inclusion. Thus a p-simplex of $\boxed{V}$ is a chain $0 < W_0 < \dots < W_p < V$ of proper subspaces W_i of V. If $n \leq 1$ then $\boxed{V} = \emptyset$; if $n = 2$ then $\boxed{V}$ is the projective space $\mathbb{P}(V)$ of lines in V, as a discrete space.

THEOREM 2 **(Solomon-Tits, cf. [Solomon]):** Suppose $n \geq 2$. Then $\boxed{V}$ has the homotopy type of a bouquet of $(n-2)$-spheres.

We shall give a proof of this below, since some of the details are needed for other arguments.

It follows that $\boxed{V}$ has zero reduced (integral) homology except in dimension $n - 2$, where we obtain a free $\mathbb{Z}$-module $H_{n-2}(\boxed{V})$ on which $GL(V)$ naturally acts. This is called the Steinberg module of V, here denoted $st(V)$. For $n = 1$ we agree to put $st(V) = \mathbb{Z}$, with trivial action of $GL(V)$.

Now let A be a Dedekind ring with field of fractions F. For each $n \geq 0$ let Q_n denote the full subcategory of $Q = Q\underline{\underline{P}}(A)$ whose objects are the projective A-modules P of rank $\leq n$. Thus Q_0 is equivalent to the trivial category, $Q_n \subset Q_{n+1}$, and $Q = \cup_n Q_n$. The main result to be proved below is the following.

THEOREM 3: Let $n \geq 1$. The inclusion $w: Q_{n-1} \to Q_n$ induces a long exact sequence

$$\cdots \longrightarrow H_i Q_{n-1} \longrightarrow H_i Q_n \longrightarrow \coprod_{\alpha} H_{i-n}(GL(P_\alpha), st(V_\alpha)) \longrightarrow H_{i-1} Q_{n-1} \longrightarrow \cdots,$$

where the P_α's represent the isomorphism classes of projective A-modules of rank $= n$, and where $V_\alpha = P_\alpha \otimes_A F$.

Remarks: (1) For any (essentially) small category C we put $H_* C = H_*(BC, \mathbb{Z})$, where BC is the classifying space

(= geometric realisation of the nerve) of C in the sense of [Quillen, 1, §1].

(2) The structure theory of projective modules over Dedekind rings implies that $P_\alpha \mapsto \det P_\alpha = \Lambda^n P_\alpha$ defines a bijection from the set of P_α's to Pic(A).

(3) The proof we give of Theorem 3 applies, more generally, when A is a maximal order, over a Dedekind domain, in a division algebra F. The corollaries of the theorem drawn below likewise apply in that generality.

COROLLARY ("Stability"). <u>The homomorphisms</u> $H_i Q_n \to H_i Q_{n+1}$ <u>are surjective for</u> $n \geq i$ <u>and injective for</u> $n \geq i + 1$.

<u>Proof of Theorem 1 from Theorem 3.</u>

Suppose now that F is a finite dimensional division algebra over $\mathbb{Q}$ and that A is a maximal order in F.

Let $P \in \underline{P}(A)$ and put $V = P \otimes_A F = P \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\Gamma = GL(P)$ is an arithmetic subgroup of $G(\mathbb{Q})$, where G is the reductive algebraic group over $\mathbb{Q}$ whose rational points in a $\mathbb{Q}$-algebra R form the group $G(R) = GL(V \otimes_{\mathbb{Q}} R)$ of $F \otimes_{\mathbb{Q}} R$-automorphisms of $V \otimes_{\mathbb{Q}} R$. Let S denote the connected component of the kernel of the norm homomorphism $G \to \mathbb{G}_m$ (the norm being that of the $\mathbb{Q}$-algebra $\mathrm{End}_F(V)$). Then S is a connected reductive algebraic group, with no non-trivial characters, defined over $\mathbb{Q}$, and so it is subject to the result of

[Borel-Serre]. Since $\Gamma \cap S(\mathbb{Q})$ has finite index in Γ (the elements of Γ having norm ± 1) we can find a normal torsion free subgroup Γ' of Γ of finite index in $\Gamma \cap S(\mathbb{Q})$. According to Théorème 3 of [Borel-Serre] we have for any Γ'-module M and any i, a duality isomorphism

$$H_i(\Gamma', I \otimes M) \cong H^{d-\ell-i}(\Gamma', M).$$

Here ℓ denotes the $\mathbb{Q}$-rank of S, d the dimension of $S(\mathbb{R})$ modulo a maximal compact subgroup, and I is the Steinberg module of the Tits building T whose simplices correspond to the parabolic subgroups of S defined over $\mathbb{Q}$. There is a natural isomorphism, in the present case, $\boxed{V} \to T$, such that the simplex $W_0 < \cdots < W_p$ of $\boxed{V}$ corresponds to its stabilizer in S, which is a parabolic subgroup defined over $\mathbb{Q}$. This isomorphism permits us to identify I with st(V), and so deduce isomorphisms

(1) $$H_i(\Gamma', st(V)) \cong H^{d-\ell-i}(\Gamma', \mathbb{Z}).$$

Now according to [Raghunathan, Cor. 3] the groups $H^j(\Gamma', M)$ are finitely generated for all j whenever M is finitely generated over $\mathbb{Z}$. Actually [Raghunathan] does not apply directly here because S is not semi-simple. However there is an exact sequence $1 \to \Gamma_s \to \Gamma' \to \Gamma_t \to 1$ where Γ_s is arithmetic in a semi-simple group and where Γ_t is finitely generated abelian. Then the groups $H^q(\Gamma_s, M)$ are finitely generated by [Raghunathan] so the spectral sequence $H^p(\Gamma_t, H^q(\Gamma_s, M)) \Rightarrow H^{p+q}(\Gamma', M)$ gives finite generation of the latter. Taking $M = \mathbb{Z}$ we obtain from (1) the finite generation of the groups $H_i(\Gamma', st(V))$. Then the homology spectral sequence $H_p(\Gamma/\Gamma', H_q(\Gamma', st(V))) \Rightarrow H_{p+q}(\Gamma, st(V))$ yields, since Γ/Γ' is finite, the finite generation of $H_i(\Gamma, st(V))$, which we now record:

(2) *If* $P \in \underline{\underline{P}}(A)$ *and* $V = P \otimes_A F$ *then* $H_i(GL(P), st(V))$ *is a finitely generated group for all* i.

The Jordan-Zassenhaus theorem (see, for example, [Bass, Ch. X, Thm. (2.4)]) implies that the set $\{P_\alpha\}$, representing isomorphism classes of projective A-modules of rank n, is finite. Hence by (2), the groups

$$L_i = \coprod_\alpha H_i(GL(P_\alpha), st(V_\alpha)),$$

where $V_\alpha = P_\alpha \otimes_A F$, are finitely generated. For $n \geq 1$, Theorem 3 furnishes a long exact sequence

$$\cdots \longrightarrow L_{i+1-n} \rightarrow H_i Q_{n-1} \longrightarrow H_i Q_n \longrightarrow L_{i-n} \longrightarrow \cdots$$

Since $H_0Q_0 = \mathbb{Z}$ and $H_iQ_0 = 0$ for $i > 0$ we conclude by induction on n that

(3) H_iQ_n <u>is finitely generated for all</u> i <u>and</u> n.

Fixing i and letting $n \to \infty$ we obtain the finite generation of $H_iQ\underline{\underline{P}}(A)$.

Now $\oplus$: $\underline{\underline{P}}(A) \times \underline{\underline{P}}(A) \to \underline{\underline{P}}(A)$ gives to $BQ\underline{\underline{P}}(A)$ the structure of a homotopy associative and commutative H-space. Hence the finite generation of its homology implies that of its homotopy, whence $K_iA = \pi_{i+1}(BQ\underline{\underline{P}}(A),0)$ is finitely generated, thus proving Theorem 1.

§2 <u>The Solomon-Tits theorem</u>.

We fix here a division ring F and a (right) vector space V of dimension $n < \infty$ over F. Let (V) denote the simplicial complex associated to the set of <u>all</u> subspaces of V, ordered by inclusion. It is contractible since, for example, V has a least subspace 0 [Quillen, 1, §1, Cor. 2 of Prop. 2]. Its p-simplices are chains

(1) $$W_0 < \dots < W_p$$

of subspaces. We distinguish the following subcomplexes of $\textcircled{V}$ by the indicated restrictions on their simplices (1):

$$\boxed{V} \quad : W_0 \neq 0 \quad \text{and} \quad W_p \neq V$$

$$\overset{\triangle}{\boxed{V}} \quad : W_0 \neq 0$$

$$\underset{\triangledown}{\boxed{V}} \quad : W_p \neq V$$

$$\overset{\triangle}{\underset{\triangledown}{\boxed{V}}} \quad : \dim(W_p/W_0) < n$$

It is readily seen that, for $n \geq 2$,

$$\overset{\triangle}{\boxed{V}} \simeq \text{Cone } \boxed{V} \simeq \underset{\triangledown}{\boxed{V}}$$

and

$$\overset{\triangle}{\underset{\triangledown}{\boxed{V}}} \simeq \text{Susp } \boxed{V} ,$$

where "$\simeq$" denotes homotopy equivalence.

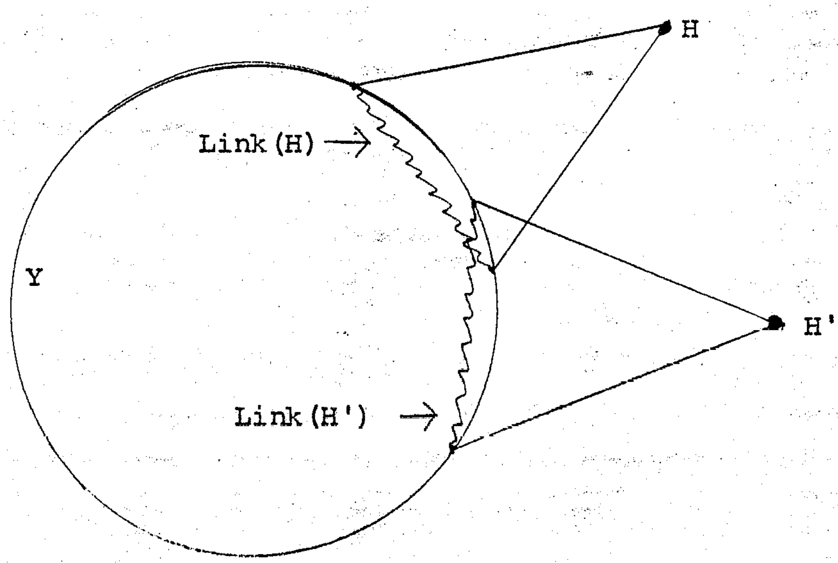
<u>Proof of Theorem 2</u>:

We argue by induction on $n \geq 2$. For $n = 2$ the discrete space $\boxed{V}$ is trivially a bouquet of 0-spheres, so assume $n \geq 3$. Fix a line L in V and let $\underline{\underline{H}}$ denote the set of hyperplanes H of V complementary to L: $V = H \oplus L$. Let Y denote the full subcomplex of $\boxed{V}$ obtained by deleting the set of vertices $\underline{\underline{H}}$.

<u>Claim</u>. Y is contractible.

In fact, let $q: V \to V/L$ be the canonical projection. It induces a simplicial map $q: Y \to \boxed{V/L}$, and the latter cone is contractible. Hence it suffices, by [Quillen, 1, §1, example following Theorem A], to show that $q^{-1}(\sigma)$ is contractible for each closed simplex $\sigma = (W_0/L < \ldots < W_p/L)$ of $\boxed{V/L}$. If U is a vertex of $q^{-1}(\sigma)$ then, for some i, we have $qU = W_i/L$, i.e. $U + L = W_i$. Thus $U \mapsto U + L$ defines a simplicial map from $q^{-1}(\sigma)$ to the simplex with vertices $W_0 < \ldots < W_p$, and the relations $U \leq U + L$ show that this map is homotopic to the identity of $q^{-1}(\sigma)$ [Quillen, 1, §1, Prop. 2]. This proves the claim.

Now we have the following schematic picture of $\boxed{V}$:



Let $H \in \underline{\underline{H}}$. Let Link(H) denote the subcomplex of $\boxed{V}$ formed by simplices σ such that $H \notin \sigma$ but $\sigma \cup \{H\}$ is a simplex. Evidently Link(H) $\subset Y$. Further $\boxed{V}$ is the union of Y with the cones over these links, amalgamated along the links, as H varies over $\underline{\underline{H}}$. From the claim above we thus obtain

$$\boxed{V} \simeq \boxed{V}/Y$$

$$\simeq \bigvee_{H \in \underline{\underline{H}}} \text{Susp (Link(H))}.$$

But clearly Link(H) $= \boxed{H}$ for $H \in \underline{\underline{H}}$. Thus the theorem follows by induction, since $\dim H = n - 1$

Let J(V) denote the set of proper layers (W_0, W_1) of V $(0 \leq W_0 \leq W_1 \leq V$ and $\dim(W_1/W_0) < n)$, ordered by $(W_0, W_1) \leq (W_0', W'_1)$ if $W_0' \leq W_0$ and $W_1 \leq W_1'$. For $n = 1$ it is the "unrelated" set consisting of (0,0) and (V,V), whence $BJ(V) = S^0$.

PROPOSITION: <u>Suppose</u> $n \geq 2$. <u>There is a</u> GL(V)-<u>equivariant homotopy equivalence</u>

$$\widehat{\boxed{V}} \longrightarrow BJ(V)$$

Define a map g from the set Simpl $\widehat{\boxed{V}}$ of simplices of $\widehat{\boxed{V}}$, ordered by inclusion, to J(V), by

$$g(W_0 < \dots < W_p) = (W_0, W_p)$$

Clearly g is order preserving, i.e. a functor (where we view ordered sets as categories) and it is also $GL(V)$-equivariant. Since $B\,\mathrm{Simpl}(K)$ is canonically homeomorphic to the barycentric subdivision of K, for any simplicial complex K (cf. [Quillen, 1, §1]) the proposition will follow if we show that Bq is a homotopy equivalence. For this it suffices, by [Quillen, 1, §1, Theorem A], to show, for each $(U_0,U_1) \in J(V)$, that the category $g/(U_0,U_1)$ is contractible (i.e. that its classifying space is so). The objects of $g/(U_0,U_1)$ are simplices $W_0 <\ldots< W_p$ such that $U_0 \leq W_0$ and $W_p \leq U_1$; they are ordered by inclusion. Evidently $g/(U_0,U_1)$ is isomorphic to Simpl (U_1/U_0) so indeed it is contractible.

<u>Remark</u>: The above proposition, and its proof, are purely combinatorial, in the following sense. Let S be a partially ordered set with a least element, 0, and a greatest element, V. Set $S' = S - \{0,V\}$, and assume $S' = \emptyset$. Then $BS \cong \mathrm{Susp}(BS')$. Let

$$J = \{(W_0,W_1) \in S \times S \mid W_0 \leq W_1, \quad \text{and} \quad 0 < W_0 \quad \text{or} \quad W_1 < V\}$$

Define $g\colon \mathrm{Simpl}(BS) \to J$ by $g(W_0 <\ldots< W_p) = (W_0,W_p)$. Then $g/(U_0,U_1) = \mathrm{Simpl}(B[U_0,U_1])$, where $[U_0,U_1] = \{W \in S \mid U_0 \leq W \leq U_1\}$. Moreover $B[U_0,U_1]$ is contractible since

$[U_0,U_1]$ has a least (and greatest) element. Hence Bg furnishes an Aut(S)-equivariant homotopy equivalence BS $\to$ BJ.

COROLLARY: Suppose $n \geq 1$. The reduced homology $\tilde{H}_i(J(V))$ ($= \tilde{H}_i(BJ(V),\mathbb{Z})$) vanishes for $i \neq n-1$. The $\mathbb{Z}$-module $\tilde{H}_{n-1}(J(V))$ is free.

DEFINITION: We call $\tilde{H}_{n-1}(J(V))$, with the natural action of GL(V) on it, the Steinberg module of V, and denote it st(V).

In view of the proposition this definition accords with that given in §1 above.

§3. Proof of the main theorem (theorem 3).

We begin by recalling some basic facts about the homology, $H_iC = H_i(BC,\mathbb{Z})$, of a small category C. A reference for this is [Gabriel-Zisman, Appendix II, §3] (cf. also [Quillen, 1, §1]).

Consider the abelian category C-Ab of abelian group valued functors on C. The functor $\underrightarrow{\lim}$: C-Ab $\to$ Ab is right exact and has left derived functors $\underrightarrow{\lim}^C_p$: C-Ab $\to$ Ab. For the constant functor $\mathbb{Z}$ we have

(1) $$H_pC = \underrightarrow{\lim}^C_p \mathbb{Z}$$

[Gabriel-Zisman, App. II, 3.3].

For example suppose C is a group G, viewed as a category with one object. A functor $M: G \to Ab$ is just a G-module, and we have $\varinjlim^G M = H_0(G,M) = M/(\Sigma_{g \in G}(g-1)M)$. Similarly

$$(2) \qquad \varinjlim{}^G_p M = H_p(G,M),$$

the Eilenberg-MacLane homology of G.

Suppose $w: C' \to C$ is a functor between small categories. If $P \in C$ we have the functor $i_P: w/P \to C'$ sending (P',u) to P'. Let $f: C' \to Ab$ be a functor. Then [Gabriel-Zisman, App. II, Thm. 3.6 and Remark 3.8] there is a spectral sequence

$$E^2_{p,q} = \varinjlim{}^C_p (P \longmapsto \varinjlim{}^{(w/P)}_q f \circ i_P) \Longrightarrow \varinjlim{}^{C'}_{p+q} f.$$

For the constant functor $f = \mathbb{Z}$ this takes the form (using (1)),

$$(3) \qquad E^2_{p,q} = \varinjlim{}^C_p (P \longrightarrow H_q(w/P)) \Longrightarrow H_{p+q}(C').$$

Now let A be a Dedekind ring with field of fractions F, as in Theorem 3. Our arguments do not require these data to be commutative, so we may, more generally, allow A to be a maximal order (over a Dedekind ring) in a division algebra F. The only feature we require is that if $P \in \underline{\underline{P}}(A)$ and $V = P \otimes_A F$

then $P' \mapsto P' \otimes_A F$ defines a bijection from the set of direct summands of P to the set of all sub F-modules of V. We put $\mathrm{rk}\, P = \dim V$ and define Q_n to be the full subcategory of $\underline{\underline{QP}}(A)$ whose objects are the P's of rank $\leq n$. Applying (3) to the inclusion functor $w: Q_{n-1} \to Q_n$ $(n \geq 1)$ we obtain a spectral sequence

$$(4) \qquad E^2_{p,q} = \varinjlim{}^{Q_n}_p (P \longrightarrow H_q(w/P)) \Longrightarrow H_{p+q}(Q_{n-1}).$$

Its analysis requires first the determination of the groups $H_q(w/P)$, for $P \in Q_n$.

Recall that an object of w/P is a morphism $u: P' \mapsto P$ with $P' \in Q_{n-1}$. Up to isomorphism (over P) such an object is determined by an admissible layer (P_0,P_1) of P such that u corresponds to an isomorphism $P' \to P_1/P_0$. Thus we see that w/P is equivalent to the set J of admissible layers (P_0,P_1) of P such that $\mathrm{rk}(P_1/P_0) < n$, with the ordering $(P_0,P_1) \leq (P_0',P_1')$ if $P_0' \leq P_0$ and $P_1 \leq P_1'$.

If $\mathrm{rk}\, P < n$ then J has the maximal element $(0,P)$, so w/P is contractible [Quillen, 1, §1, Cor. 2 to Prop. 2].

Suppose $\mathrm{rk}\, P = n$. Then the map $P' \mapsto P' \otimes_A F \subset V = P \otimes_A F$ induces an isomorphism from J to $J(V)$ (notation as in §2). Thus, in view of the proposition and corollary of §2, we can now describe the groups $H_q(w/P)$.

For $n = 1$ we have

$$H_q(w/P) = 0 \qquad \text{if } q > 0$$

(5)
$$H_0(w/P) = \begin{cases} \mathbb{Z} & \text{if } P = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if rk } P = 1. \end{cases}$$

For $n \geq 2$ we have

(6)
$$H_0(w/P) = \mathbb{Z}$$
$$H_q(w/P) = 0 \qquad \text{if } q \neq 0,\ n-1$$
$$H_{n-1}(w/P) = \begin{cases} 0 & \text{if rk } P < n \\ \mathrm{st}(V) & \text{if rk } P = n \end{cases}$$

Consider first the case $n \geq 2$. Then

$$\varinjlim{}_P^{Q_n} (P \longrightarrow H_q(w/P)) = \begin{cases} H_p(Q_n) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0,\ n-1 \end{cases}$$

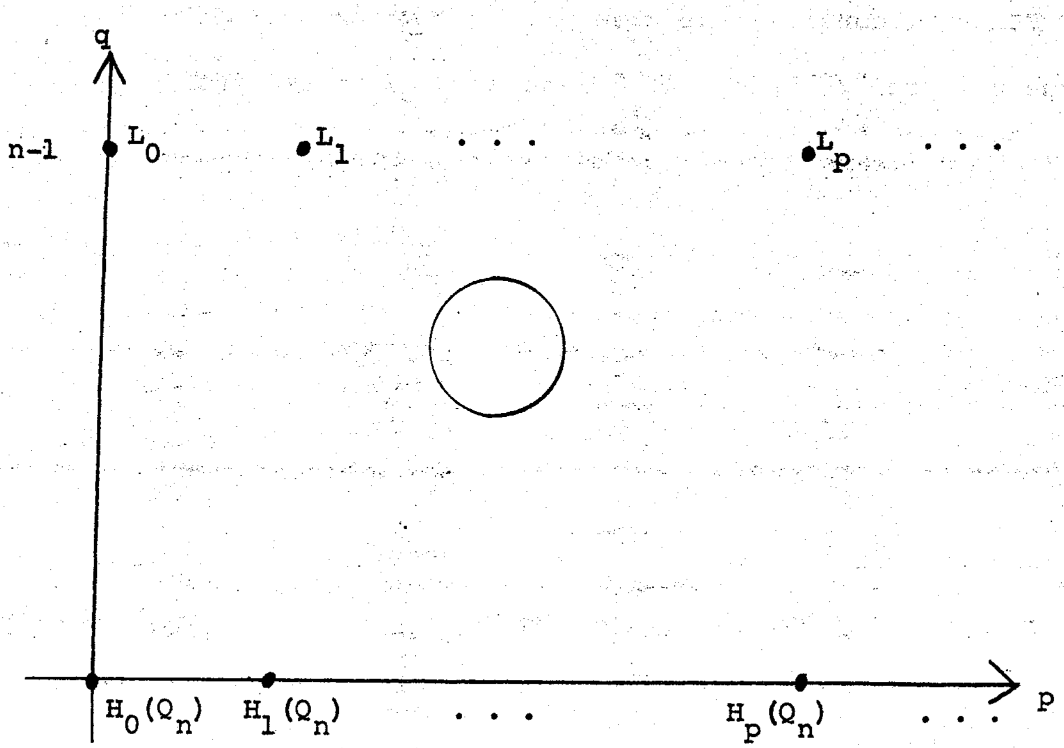
For the case $q = n - 1$ we introduce the full subcategory Q' of Q_n whose objects are the P's of rank $= n$. Then $h(P) = H_{n-1}(w/P)$ is, by (6) a functor in Q_n which vanishes on Q_{n-1}, and hence on the source of every arrow whose target lies outside Q'. It follows that the complex in [Gabriel-Zisman, App. II, 3.2] use to compute $\underset{\to}{\lim}{}_*^{Q_n} h$ is isomorphic to that used to compute $\underset{\to}{\lim}{}_*^{Q'} (h|Q')$. Now Q' is equivalent to the groupoid of projective modules of rank n and their isomorphisms. It is equivalent to the full skeletal subcategory with one object P_α from each isomorphism class, and the latter category is just the groupoid Q" = $\coprod_\alpha GL(P_\alpha)$. On Q" the functor $P_\alpha \mapsto H_{n-1}(w/P_\alpha)$ corresponds to the family of $GL(P_\alpha)$-modules $st(V_\alpha)$, where $V_\alpha = P_\alpha \otimes_A F$, by (6). In view of (2) therefore we have

$$\underset{\to}{\lim}{}_p^{Q_n} (P \longmapsto H_{n-1}(w/P))$$

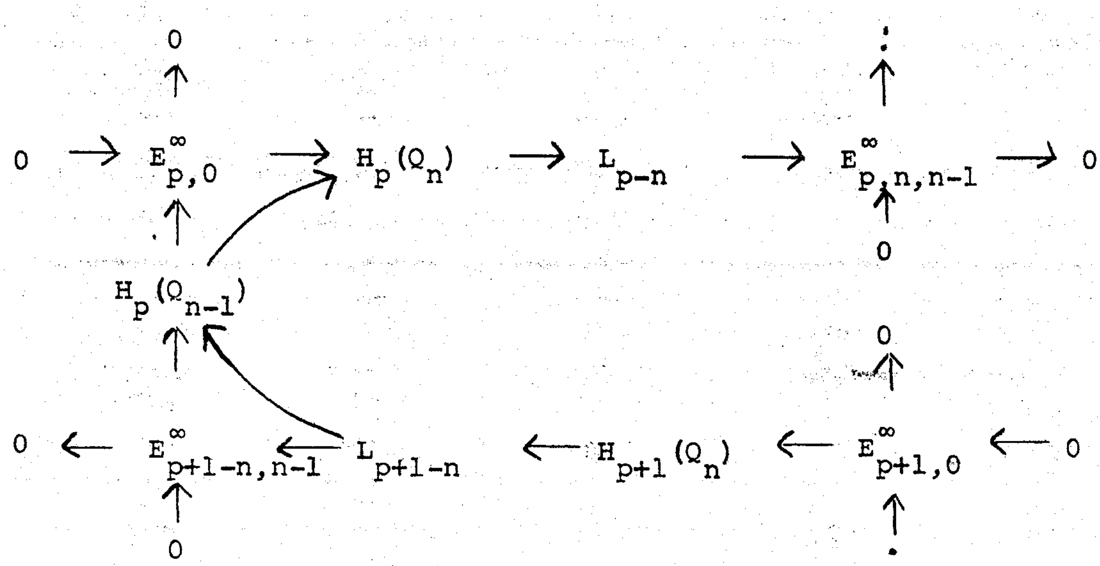
$$= \coprod_\alpha H_p(GL(P_\alpha), st(V_\alpha))$$

Denoting this group by L_p we can now display

$$E^2_{p,q} = \underset{\to}{\lim}{}_p^{Q_n} (P \longmapsto H_q(w/P)).$$



This picture remains in tact until the E^n-term, whose differentials furnish the horizontal sequences in the exact diagram



from which one extracts the exact sequence of Theorem 3.

Finally, consider the case n = 1. Then in view of (5) the spectral sequence (4) degenerates to an isomorphism

$$\underset{\rightarrow p}{\lim}^{Q_1} (P \longmapsto H_0(w/P)) \cong H_p(Q_0) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0 \end{cases}$$

If h denotes the functor $P \mapsto H_0(w/P)$ we have a functorial exact sequence

$$(7) \qquad 0 \longrightarrow \tilde{H}_0(w/P) \xrightarrow{d} H_0(w/P) \longrightarrow \mathbb{Z} \longrightarrow 0$$

where d is the antidiagonal map $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ for P of rank 1. Just as in the case $n \geq 2$ above we find that

$$\underset{\rightarrow p}{\lim}^{Q_1} (P \longmapsto \tilde{H}_0(w/P)) = \coprod_{\alpha} H_p(GL(P_\alpha), st(V_\alpha))$$

a module we denote by L_p. (Recall that $st(V_\alpha) = \mathbb{Z}$ here since n = 1.) The homology exact sequence of $\underset{\rightarrow}{\lim}_*^{Q_1}$ for (7) thus takes the form

$$\cdots \longrightarrow H_p(Q_0) \longrightarrow H_p(Q_1) \rightarrow L_{p-1} \rightarrow H_{p-1}(Q_0) \rightarrow \cdots,$$

thus proving Theorem 3 for n = 1, and so completing its proof.

Orders in semi-simple algebras.

To apply this theorem to a maximal order A in a semi-simple algebra we first note that A is a finite product of maximal orders A_j in simple algebras, and K_iA is the direct sum of the K_iA_j's. Further each A_j is of the form $\mathrm{End}_{B_j}(P_j)$ where B_j is a maximal order in a division algebra and P_j is a faithful projective left B_j-module. Then $P \mapsto P \otimes_{B_j} P_j$ is an equivalence of categories $\underline{\underline{P}}(B_j) \to \underline{\underline{P}}(A_j)$, whence isomorphisms $K_iB_j \to K_iA_j$. Since Theorem 3 above applies to the B_j's we can use it to obtain information about K_iA, for example that K_iA is finitely generated when A is a maximal order in a semi-simple $\mathbb{Q}$-algebra.

In case A is a not necessarily maximal order we can embed A in a maximal order B. Then $A[\frac{1}{s}] = B[\frac{1}{s}]$ for some central non divisor of zero s (in say the conductor, i.e. the annihilator of B/A). Then the localisation sequence takes the form $.. \to K_i(\mathcal{M}) \to K_iA \to K_iB[\frac{1}{s}] \to ..$ where $\mathcal{M}$ is the category of s-torsion finitely generated A-modules of finite homological dimension. Since $B[\frac{1}{s}]$ is a maximal order the groups $K_iB[\frac{1}{s}]$ can be treated by the methods above. Additional techniques seem to be required for the analysis of $K_i\mathcal{M}$.

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