18.700 A FEW REMARKS ON MATHEMATICAL WRITING

This note is meant to correct a few of the flaws and bad habits which the graders have noticed on the problem sets. This is not meant to be a guide to mathematical writing. Having said that, I encourage you all to get a copy of Prof. Kleiman's guide [1] to mathematical writing (available outside his office 2-278). It is primarily aimed at students who are writing their math phase two paper. But many of the suggestions, particularly those in section 4, apply just as well to writing math problem sets.

1. Use words to explain your equations, computations and diagrams. This is the most common problem students have when they begin to write mathematical arguments. Before you write any formula down, read it to yourself in your mind. Your first instinct should be to write down all the words you said in your mind, with the exception of words that you read to yourself in symbol form. For example, consider the formula:

$$\forall v \in \mathbb{F}^n, \exists w \in \mathbf{F}^n, v + w = 0. \tag{1}$$

If you are explaining this sentence to a friend at a blackboard, this formula is perfectly acceptable. But if you are writing a mathematical argument to be read by other people, it is preferrable to write:

For every vector v in $\mathbb{F}^n$, there exists a vector w in $\mathbb{F}^n$ such that $v + w$ equals zero.

2. Explain all the steps in your argument. The steps you use to go from one formula to another may make sense to you. But you have been thinking about the problem you are solving for a while before you write your solution. The ideas which are clear to you at the end of this process will usually not be clear to your reader. Even if you think the argument will be clear, err on the side of writing too much. It may seem pedantic to you, and later in your mathematical career you may return to writing arguments in which small steps are left to the reader. But when you first start writing mathematics, you need to get in the habit of explaining every step that it is not completely obvious. For example, consider the following part of a proof of exercise 3 from the second homework:

$$\begin{aligned} (Av_1, \ldots, Av_r) &\ \textit{linearly independent}, \\ c_1 v_1 + \ldots c_r v_r &= 0, \\ c_1(Av_1) + \cdots + c_r(Av_r) &= 0, \\ c_1 = \cdots = c_r &= 0. \end{aligned}$$

If you were talking through this argument with a friend at a blackboard, this very likely would be sufficient to convey the argument. But a clear and complete argument when written would look very different:

Proof. Suppose that $(Av_1), \ldots, (Av_r)$ is a linearly independent collection of vectors in $\mathbb{F}^m$. We will prove that $v_1, \ldots, v_r$ is a linearly independent collection of vectors in $\mathbb{F}^n$.

In order to prove that the collection of vectors $(v_1, \ldots, v_r)$ in $\mathbf{F}^n$ is linearly independent, we must show that the only linear relation among these vectors is the trivial relation. Suppose that we have a linear relation

$$c_1 v_1 + \cdots + c_r v_r = 0. \tag{2}$$

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If we multiply both sides of this equation on the left by the $m \times n$ matrix A, we obtain the formula

$$A\,(c_1 v_1 + \cdots + c_r v_r) = A0. \tag{3}$$

It is clear that $A0$ equals zero. By distributivity of matrix multiplication with addition, we have that

$$A\,(c_1 v_1 + \cdots + c_r v_r) = A(c_1 v_1) + \ldots A(c_r v_r). \tag{4}$$

Since matrix multiplication commutes with scalar mutliplication, we can rewrite the right side of the last equation as

$$c_1(Av_1) + \cdots + c_r(Av_r). \tag{5}$$

Combining these formulas, we have the linear relation among the vectors $(Av_1), \ldots, (Av_r)$:

$$c_1(Av_1) + \ldots c_r(Av_r) = 0. \tag{6}$$

By assumption, the collection of vectors $(Av_1), \ldots, (Av_r)$ is linearly independent, so the linear relation above is the trivial linear relation: $c_1 = \cdots = c_r = 0$. Therefore we conclude that the only linear relation among $v_1, \ldots, v_r$ is the trivial linear relation, i.e. $v_1, \ldots, v_r$ is a linearly independent set of vectors. □

Although there are formulas above, they are used sparingly and only when writing them out does not clarify what is going on. Also note the use of words such as "since", "so", "therefore". These are keywords that let your reader know that you have made a deductive step. Also notice that not all statements are justified, e.g. the sentence, "It is clear that $A0$ equals zero." This step is obvious enough that it can be stated without justification. But whenever there is doubt in your mind about whether a statement is obvious, present the justification.

3. State all hypotheses. State them at the beginning of your argument. You will usually need to use all hypotheses of a result in the proof of the result (occasionally there will be a superfluous hypothesis, but not often). When you use a hypothesis, always call attention to this fact. But even if you use a hypothesis in the body of an argument, gather all hypotheses at the beginning of the argument to make it easier for your reader to see what conditions must hold for the validity of the argument.

4. Learn and stick to the basics of logic. The "basics of logic" are such things as "If proposition 1 holds, and if proposition 1 implies proposition 2, then proposition 2 holds as well." It may seem that such basics are self-evident, but in a long argument these rules occur so often that it is easy to make a mistake if one isn't diligent. One very common mistake on the homework occurred in problem 2, part (b). You are asked to prove that the set of vectors produced by the algorithm gives a spanning set for the intersection $W_1 \cap W_2$. Many students proved that every vector in the span of the set of vectors is in $W_1 \cap W_2$. Some students proved that any vector in $W_1 \cap W_2$ is in the span of the set of vectors produced by the algorithm. In a correct argument, you must prove both that every vector in the span of the set of vectors is in $W_1 \cap W_2$, and that every vector in $W_1 \cap W_2$ is in the span of the set of vectors. You must prove both implications; proving one alone does not suffice.

5. Use definitions correctly. There are so many definitions in a math class, that it is easy to be confused. Looking up definitions is part of the "research" in "research mathematics". It may not be fun or glamorous, but it is important nonetheless. One common mistake on the homework was to confuse the definition of *linear combination* with the definition of *span*. The sentence "the vector subspace W of the vector space V is the linear combination of the vectors $v_1, \ldots, v_n$ of V" makes no sense. One cannot speak of a vector space or vector subspace as being a linear combination of vectors. The correct statement is either "the vector subspace W of the vector space V is spanned by the vectors $v_1, \ldots, v_n$ of V", or perhaps "every vector w in the vector subspace W of V is a linear combination of the vectors $v_1, \ldots, v_n$" (although this second statement allows the possibility that W is properly contained in the span of $v_1, \ldots, v_n$).

6. Do not write in "stream-of-consciousness": you should not write all the thoughts that go through your mind as you solve a problem. We do not think of ideas for ourselves in the same order that we explain them to others. Once you have developed an argument for yourself, you still need to take time and rearrange the steps in correct logical order. Also, the best order for presenting ideas in mathematical writing may not be the order which occurs to you first, even if that order is logical order. For example, suppose you were asked to give a counterexample to the statement, "Every integer is a difference of two square integers." One likely approach is to simply start considering the whole numbers $0, 1, 2$ etc. in order and see whether we can write these as a difference of squares, e.g. $0 = 0^2 - 0^2, 1 = 1^2 - 0^2$. When we get to 2, we have a problem. If we add some small squares to 2, we see that we do not get another square number. Eventually it will occur to us to rewrite the equation

$$2 = a^2 - b^2 \tag{7}$$

by factoring $a^2 - b^2$, i.e.

$$2 = (a - b)(a + b). \tag{8}$$

Since 2 is a prime number, the only possibilities are $a - b = 1, a + b = 2$, $a - b = -1, a + b = -2$, $a - b = 2, a + b = 1$, or $a - b = -2, a + b = -1$. We can then solve each of these 2×2 inhomogeneous systems of equations and observe that none of them has integer solutions. Thus we have found a counterexample.

But this is NOT how we should write the argument for others to read. First of all, the order is backwards: one should state a result and then give the proof, not give a sequence of steps that lead to a result only at the end. Also, there are several small simplifications we can make that will save the reader time – the burden is on the author of an argument to check details and simplify arguments. One possibly solutions of the problem above is the following.

Proof. The integer 2 cannot be expressed as a difference of two square integers. We will prove this by contradiction. By way of contradiction, suppose there exist integers a and b such that $2 = a^2 - b^2$. We may factor the right-hand side of this equation as $2 = (a - b)(a + b)$. Since 2 is a prime number, either 2 divides $a - b$ or 2 divides $a + b$.

Consider first the possibility that 2 divides $a - b$. Either $a - b = 2$ or $a - b = -2$, and in these cases we also have that $a + b = 1$ or $a + b = -1$ respectively. We may rewrite the first pair of equation as $a = b + 2$ or $a = b - 2$ respectively. Plugging this in to the second pair of equations, we conclude that $2b + 2 = 1$ or $2b - 2 = -1$ respectively. But each of these equations can be factored as $2(b + 1) = 1$ or $2(b - 1) = -1$ respectively. The first equation implies that 2 divides 1, and the second equation implies that 2 divides -1. Both of these statements are absurd, thus our hypothesis is false: 2 does not divide $a - b$.

The second possibility is that 2 divides $a + b$. Then either $a + b = 2$ or $a + b = -2$, and in these cases we also have that $a - b = 1$ or $a - b = -1$ respectively. We may rewrite the first pair of equations as $a = -b + 2$ or $a = -b - 2$ respectively. Plugging this into the second pair of equations, we conclude that $-2b + 2 = 1$ or $-2b - 2 = -1$. But each of these equations can be factored as

$2(-b+1) = 1$ or $2(-b-1) = -1$ repsectively. The first equation implies that 2 divides 1, and the second equation implies that 2 divides -1. Both of these statements are absurd, thus our hypothesis is false: 2 does not divide $a+b$.

Since both the possibility that 2 divides $a-b$ and the possibility that 2 divides $a+b$ are proved false, our original hypothesis must also be false. In other words, there is no pair of integers a and b such that $2 = a^2 - b^2$. □

Notice in particular that we do not "leave it to the reader" to check that $2 = (a-b)(a+b)$ has no solution, even though it is easy to check this. Later on in your mathematical career, you will likely sometimes leave easy calculations to your reader, but in this class you must check all details which are not obvious.

7. If you are using a "proof by contradiction" or a "proof by induction", say you are doing so at the beginning of the argument. In a proof by contradiction, use the keyphrase, "By way of contradiction let us assume . . ." At the end of a proof by contradiction, after you have derived an absurdity by assuming the negation of what it is you want to prove, you still have something to say. Use a couple of sentences such as, "This result is absurd. Therefore we conclude that our original hypothesis is false, which is to say . . ." and then state the result you wanted to prove. A proof by contradiction is only complete after you have explained how the absurdity leads to the statement you wanted to prove.

In a proof by induction, always remember the *base case*, or the step "n equals 1" (or possibly n equals 0 or some other integer depending on the exact statement you want to prove). Present the base case first, before the induction step. After you have proved the base case, state the induction hypothesis and say you are doing so. Use a sentence like, "By way of induction, let us suppose that the result is known for the integer n." Explain how using the hypotheses of the result you are proving and using the induction hypothesis, you conclude the result for $n+1$. Call attention to exactly where in the argument you use the induction hypothesis. If you never use the induction hypothesis, you should not be presenting the argument as a proof by induction (even if it is logically valid to do so, it only confuses the reader to present a direct argument of a result as a proof by induction).

Some mathematicians (admittedly a small minority) reject proof by contradiction. Even were there no such mathematicians, as a matter of style, you should not use a proof by contradiction where a direct proof will do. Some people use proof by contradiction as a labor-saving device: They will state all hypotheses, assume the negation of this collection of hypothesis, write down alot of steps (sometimes incoherent, often difficult to follow) which eventually lead to an absurdity, and then claim to have given a valid argument. This is unacceptable: remember the responsibility for doing the work of the argument and presenting a clear proof is on the author, not the reader. Sometimes a proof by contradiction is unavoidable; only use it in these cases.

These are some of the rules of style you should use in writing mathematical arguments. There are many others. The best way to learn these rules is to read math proofs: both good and bad. Good math proofs are easy to find; just pick up most textbooks or math journals (you do not need to understand the mathematics involved to appreciate the style of a well-written argument). Bad proofs, unfortunately are also easy to find. It is a matter of patience, practice and experience to produce good math arguments and not bad ones.

References

[1] Steven L. Kleiman with the collaboration of Glenn P. Tesler, *Writing a Math Phase Two Paper*, copyright January 4, 2000.