MATH 402, HOMEWORK 2

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Problems on Galois theory

Problem 1 (10 points). Show that any 5-cycle and any 2-cycle in the symmetric group S_5 generate S_5, and use that to find the Galois group of $x^5 - 20x^3 + 2$ over $\mathbb{Q}$.

Problem 2 (10 points). Let K/F be a Galois extension with $G = \mathrm{Gal}(K/F) = \{\sigma_1, \ldots, \sigma_n\}$ and with a basis $\{y_1, \ldots, y_n\}$ for K over F. Consider the n by n matrix A where $A_{ij} = \sigma_i(y_j)$. We showed in class that A is nonsingular. Show that $D = (\det A)^2 \in F$. Show that modifying the choice of how to number the elements of G or modifying the choice of the basis for K changes D at most by multiplying it by the square of an element of F.

Problem 3 (10 points). Find the Galois group over the rational numbers of $x^4 - 4x + 2$.

Problem 4 (10 points). Give an example of field extensions $F \subseteq E \subseteq K$ so that E is Galois over F and K is Galois over E but K is not Galois over F.

Problem 5 (10 points). Let ζ be a primitive 9-th root of unity in $\mathbb{C}$. What group is $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$? (Describe it concretely in terms of generators and how they act on the field, and abstractly, by finding an isomorphism with a group we've seen before.) Show the only proper intermediate fields are $\mathbb{Q}(\zeta + \zeta^{-1})$ and $\mathbb{Q}(\zeta^3)$, and describe the corresponding subgroups of G by giving generators for them. Find the minimal polynomials over $\mathbb{Q}$ for ζ, $\zeta + \zeta^{-1}$, and ζ^3.

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Date: February 10, 2000, due February 20.