Problem 7.

Suppose p is a prime number, G is a finite group of order n, and p^r is the largest power of p that divides $|G|$. Suppose $\alpha \leq r$, and let X be the set of subsets $T \subseteq G$ with $|T| = p^\alpha$. Let G act on X by defining $g \cdot T = \{gt \mid t \in T\}$.

(a) Show the stabilizer G_T acts on the set T. Show that T is a union of cosets of G_T.

(b) If $T \in X$ were a subgroup, what would the cardinality of its orbit $G \cdot T \subseteq X$ be? What if T were a coset of some subgroup? What if T were some other subset?

(c) What is the largest power p^β of p dividing $|X|$?

(d) Show there is an orbit $G \cdot x$ whose cardinality is not divisible by $p^{\beta+1}$.

(e) Compute the order of the stabilizer G_x.

Problem 7 continued.

Let Y be the set of subgroups of G of cardinality p^r. Let $n_p = |Y|$.

(a) Let G act on Y by conjugation. Show any orbit Z has cardinality not divisible by p.

(b) Suppose H is a subgroup of G of order p^α. Let H act on on Z by conjugation. Show there is a fixed point. Show H is contained in an element of Z.

(c) Show any two subgroups of order p^r are conjugate.

(d) Show that n_p divides $|G|$.

(e) Show $n_p \equiv 1 \pmod{p}$.